Learning and Working With Generalized Functions Can Be Fun

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Outline of the Talk

– What is an ordinary function?
– What is a generalized function (GF)?
– How do we extend the definition of a function?
– A brief definition of GFs
– Differentiation of GFs
– Some applications of GFs
– Concluding remarks
What is an Ordinary Function (OF)?

We think of an ordinary function such as $\sin x$ or $e^x$ as a table of ordered pairs of numbers $(x, \sin x)$ or $(x, e^x)$. This means that we think as ordinary functions pointwise. A graph is a way is illustrating this pointwise view of a function.

- The functions engineers learn in calculus are all ordinary functions.
- These functions are not enough for what the engineers need later.
What is a Generalized Function (GF)?

The British physicist Dirac in his work on quantum mechanics in 1926 introduced a “function” called delta function with the property

\[
\int_{-\infty}^{\infty} \varphi(x) \delta(x) dx = \varphi(0)
\]

We can show that there is no ordinary function with this property. However, this “function” is so important in mathematics that mathematicians had to find a way to define it rigorously. It took about 25 years to do this!
How Do We Extend the Definition of a Function?

We must change the way we think of functions from pointwise (ordered pair of numbers) to one that allows new objects or functions. This process of extension of the definition of ordinary functions is difficult and most mathematicians never have to do this.

People involved: Sobolev, Schwartz, Mikusinski
How Do We Extend the Definition of a Function? (Cont’d)

When we extend the definition of a function, we require two things:

– All ordinary functions must be included in the extension

– All the operations on ordinary functions must also apply to the new functions

The most important thing to know about extension of ordinary functions to GFs is that we can solve new problems of engineering and science.
A Brief Definition of GFs

We take some well-behaved functions which we call test functions (TFs). For an OF \( f(x) \), we define the following mapping (functional) of TFs to real numbers

\[
F[\varphi] = \int_{-\infty}^{\infty} f(x)\varphi(x)dx, \quad \varphi(x) \text{ is a TF}
\]

This mapping is linear:

\[
F[\alpha\varphi_1 + \beta\varphi_2] = \alpha F[\varphi_1] + \beta F[\varphi_2]
\]

and continuous:

\[
F[\varphi_n] \to 0 \quad \text{if} \quad \varphi_n \to 0
\]
Models of Functions

Old (Conventional) Model: We think of a function as a table of ordered pairs \((x, f(x))\) where for each \(x, f(x)\) is unique. This table can be graphed as shown and usually has an uncountable number of ordered pairs.

New Model: We think of a function \(f\) by its action (functional values) on a given space of ordinary functions called test function space. This action for ordinary functions is defined by

\[
F[\phi] = \int f(x)\phi(x)\,dx.
\]

The function \(f\) is now defined (identified, thought of) by the new table \(\{F[\phi], \phi \text{ is in the test function space}\}\). This view of ordinary functions now allows us to incorporate \(\delta(x)\) into mathematics rigorously.
A Familiar Example of Thinking About Functions by New Model

Consider space of periodic functions with period $2\pi$. Take the test function space to be the space formed by functions $\phi_n = \exp(inx)$, $n = 0, \pm 1, \pm 2, \ldots$. Let $f$ be periodic with period $2\pi$. The Fourier coefficients of $f$ can be viewed as functionals on test function space by the relation

$$F[\phi_n] = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{inx} \, dx$$

From the theory of Fourier analysis, we know that the following table of Fourier coefficients (i.e., functional values of $f$ on test function space) contains the same information as $f(x)$:

$$\{ F[\phi_n], n = 0, \pm 1, \pm 2, \ldots \}$$

Note that if $f(x) \neq g(x)$, where $g(x)$ is another periodic function with period $2\pi$, then

$$F[\phi_n] \neq G[\phi_n] = \frac{1}{2\pi} \int_0^{2\pi} g(x)e^{inx} \, dx$$

for some $n$, i.e., the new table uniquely defines functions.
We define GFs as all the continuous linear functionals on some TF space.

- All ordinary functions are GFs

Example: Dirac delta function

Define this GF by the functional \( \Delta[\varphi] = \varphi(0) \) which can be shown to be linear and continuous. Now we introduce a symbolic function \( \delta(x) \) (i.e., looking like an OF but is not an OF) where \( \Delta[\varphi] = \varphi(0) \) is written as

\[
\int_{-\infty}^{\infty} \delta(x)\varphi(x)dx.
\]

Now work with \( \delta(x) \) like an OF!
Definition of Generalized Functions (Cont’d)

• It is inconvenient to work with functional notation in mathematical manipulations. For this reason, we introduce the notation of symbolic functions for those generalized functions which are not ordinary functions. Ordinary functions are called *regular* generalized functions. Other generalized functions are called *singular* generalized functions. For singular generalized function $F[\phi]$, we define the symbolic function $f(x)$ so that
\[ \int f(x)\phi(x)\,dx \equiv F[\phi] \text{ for } \phi \in D. \]
It is important to recognize that the integral on the left is just a symbol standing for $F[\phi]$ and one should not treat it as an ordinary integral.

This is the picture of the space of generalized functions $D'$ we should have in mind.
A Brief Definition of GFs (Cont’d)

– We extend all the operations on OFs to GFs and work with the notation of OFs mostly.

Examples: addition and multiplication of functions, translation, scaling, differentiation, integration, Fourier transformation, etc.

\[ \int_{-\infty}^{\infty} \varphi(x) \delta(x - x_0) dx = \varphi(x_0), \]

\[ \int_{-\infty}^{\infty} \varphi(x) \delta(\alpha x) dx = \frac{\varphi(0)}{|\alpha|}, \int_{-\infty}^{\infty} e^{2\pi i \lambda x} \delta(x) = 1 \]
Differentiation of Generalized Functions

All test functions are in D.

- \( f(x) \) ordinary function, differentiable, \( F[\phi] = \int f\phi dx \), we must identify \( F'[\phi] \) with \( \int f'\phi dx \). But \( F'[\phi] = \int f'\phi dx = -\int f\phi' dx = -F[\phi'] \) since \( \phi'\in D \). Therefore, we use the relation:

\[
F'[\phi] = -F[\phi']
\]

as the definition of derivative of any generalized function \( F[\phi] \). Similarly \( F^{(n)}[\phi] = (-1)^n F[\phi^{(n)}] \), i.e., generalized functions have derivatives of all orders.

Examples:

i) \( \delta'[\phi] = -\delta[\phi'] = -\phi'(0) \) or \( \int \delta'(x)\phi(x) dx = -\phi'(0) \)

ii) \( \delta''[\phi] = (-1)^2 \delta[\phi''] = \phi''(0) \) or \( \int \delta''(x)\phi(x) dx = \phi''(0) \)

Note: If an ordinary function is differentiable on real line, then \( f'_\text{gen.} = f' \). However, generalized derivative of an ordinary function can be a singular generalized function.
Differentiation of Generalized Functions (Cont’d)

Notation: For ordinary (regular G.F.’s) functions, we use \( \bar{f}'(x) \) or \( \frac{df}{dx} \) for \( f' \) to distinguish generalized from ordinary derivative.

Example: Generalized derivative of an ordinary function with a jump.

\[
F[\phi] = \int f\phi \, dx, \phi \in \mathcal{D}
\]

\[
F'[\phi] = -F[\phi'] = -\int f \, \phi' \, dx
\]

\[
= - \left( \int_{-\infty}^{x_0^-} + \int_{x_0^+}^{\infty} \right) f \, \phi' \, dx = \int f' \phi \, dx + \Delta f \, \phi(x_0)
\]

or symbolically

\[
\bar{f}'(x) = f'(x) + \Delta f \, \delta(x - x_0)
\]
Differentiation of Generalized Functions (Cont’d)

Example: Generalized derivative of Heaviside function

\[ h(x) = \begin{cases} 
1 & x > 0 \\
0 & x < 0 
\end{cases} \]

\[ h'(x) = \delta(x) \text{ since } h'(x) = 0 \text{ on } (0, \infty) \cup (-\infty, 0). \]

• Note: Even at this level of exposition, we can do a lot we could not do by using ordinary functions. We can discuss Green’s function of an O.D.E., for example.
Some applications of GFs

– Finding Green’s functions and solutions of ODEs and PDEs

– Deriving transport theorems of fluid mechanics, thermodynamics, etc.

– Deriving jump conditions across discontinuities from conservation laws

– Deriving mathematical results, e.g., Leibniz rule of differentiation under integral sign, Kirchhoff formula, etc.

– Deriving conservation laws

The real Power of GFs theory is revealed in multidimensional problems
Generalized Functions in Multidimensions (Cont’d)

- **Generalized functions** in \( n \) dimensions are continuous linear functionals on \( n \) dimensional test function space \( D \).

- **Examples:**
  
  i) \[ \int \delta(\hat{x})\phi(\hat{x})d\hat{x} = \phi(0) \]
  
  ii) \[ \int \left[ \frac{\partial}{\partial x_i} \delta(\hat{x}) \right] \phi(\hat{x})d\hat{x} = - \frac{\partial \phi}{\partial x_i}(0) \]

- From our point of view, the most important generalized functions are delta functions whose supports are on open or closed surfaces, e.g., \( \delta(f) \). We need to interpret integrals of the form

\[
I_1 = \int \delta(f)\phi(\hat{x})d\hat{x} \quad \text{and} \quad I_2 = \int \delta'(f)\phi(\hat{x})d\hat{x}.
\]
How Does $\delta(f)$ Appear in Applications?

Assume $g(\hat{x})$ is discontinuous across the surface $f(\hat{x}) = 0$ with the jump

$$\Delta g = g(f = 0+) - g(f = 0-)$$

Set up coordinate system $(u^1, u^2)$ on $f = 0$ and extend these coordinates to the vicinity of $f = 0$ along local normals. Take $u^3 = f$ as third local variable. Then (assuming $g$ is continuous in $u^1, u^2$)

$$\frac{\partial g}{\partial u^i} = \frac{\partial g}{\partial u^i} \quad i = 1, 2 \quad \text{and} \quad \frac{\partial g}{\partial u^3} = \frac{\partial g}{\partial u^3} + \Delta g \delta(u^3)$$

$$\frac{\partial g}{\partial x_j} = \frac{\partial g}{\partial u^i} \frac{\partial u^i}{\partial x_j} = \frac{\partial g}{\partial u^i} \frac{\partial u^i}{\partial x_j} + \Delta g \frac{\partial u^3}{\partial x_j} \delta(u^3) = \frac{\partial g}{\partial x_j} + \Delta g \frac{\partial u^3}{\partial x_j} \delta(u^3)$$

Since $u^3 = f$, we have $\nabla^\bot g = \nabla g + \Delta g \nabla f \delta(f)$.

Similarly

$$\nabla \cdot \hat{g} = \nabla \cdot \hat{g} + \Delta \hat{g} \cdot \nabla f \delta(f)$$

$$\nabla \times \hat{g} = \nabla \times \hat{g} + \Delta \hat{g} \times \nabla f \delta(f)$$
The Integration of \( \delta(f) \) and \( \delta'(f) \)

We assume \( f(\hat{x}) \) is defined such that \( |\nabla f| = 1 \) on the surface \( f = 0 \). This can always be done. This means \( df = dn = du^3 \)

- Parametrize the space in vicinity of surface \( f = 0 \) by variables \( (u^1, u^2, u^3) \) as shown. Then

\[
I_1 = \int \phi(\hat{x}) \delta(f) d\hat{x}
\]

\[
d\hat{x} = \sqrt{g(3)} du^1 du^2 du^3
\]

\[
= \sqrt{g'(2)}(u^1, u^2, u^3) du^1 du^2 du^3
\]

\[
I_1 = \int \phi(\hat{x}) \delta(u^3) \sqrt{g'(2)} du^1 du^2 du^3 = \int [\phi(\hat{x})]_{u^3} = 0 \sqrt{g(2)} du^1 du^2
\]

\[
I_1 = \int \phi(\hat{x}) \delta(f) d\hat{x} = \int_{f = 0} \phi(\hat{x}) ds
\]
Concluding Remarks

– GFs are generalization of ordinary functions with very nice properties

– GFs are very useful in many problems of engineering

– Many new problems can be solved which cannot be solved using OFs

– Learning GFs is easy even for multidimensions if abstractions are kept to minimum and operational properties are emphasized. It can actually be fun!